Galerkin approximation, dimensional reduction, heat conduction.

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A GALERKIN APPROXIMATION METHOD INCLUDING SPACE DIMENSIONAL REDUCTION - APPLIED FOR SOLUTION OF A HEAT CONDUCTION EQUATION

Abstract

A multivariate data fitting procedure, based on the Galerkin minimization method, is studied in this paper. The main idea of the developed approach consists in projecting the set of data points from the original, higher-dimensional space, onto a line section. Then, the approximation problem is solved in the resulting one-dimensional space. The elaborated recipe can be designed so that it is computationally more efficient than the schemes based on the least squares minimization. The performance of the method is studied by comparison with the least squares and the moving least squares procedures in a number of examples, including the solution of the heat diffusion equation.

1. INTRODUCTION

Numerical solution of engineering and scientific problems is most often equivalent to solution of some approximation task. In the framework of standard finite element method (FEM) this is accomplished by defining interpolation functions over local subdomains of various shapes, and these functions are frequently chosen from the space of polynomials. Examples can be found in the textbooks [1][2]. In the area of mesh-free or grid-free methods, broadly discussed in the paper [3] and subsequently, for example, in the work [4], local representation of an unknown function is commonly obtained by using the least squares or the weighted least squares fit. If the weight function is defined at each point at which the approximation is to be evaluated then such an approach is named the moving least squares (MLS) method and is thoroughly characterized in the reference [5]. Further insight into the method can be found

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in the work [6], where the analysis of error estimates is given. The benefits of the MLS fit are well reflected by its wide application range, incorporating the solution schemes for the partial differential equations.

Excellent smoothing properties of the MLS fit is occupied, however, by increased computation times, relative to the performances of the other approximation methods. Therefore, efforts are undertaken to improve its efficiency, and the work reported in [7] can serve as an example. An improvement of the MLS methodology, named the approximate MLS approximation, is also discussed in a series of papers including references [8] and [9]. This method has advantage of being matrix-free for a certain class of problems and therefore robust. Its disadvantage of being not enough exact for irregularly spaced data seems to be overcame by iterated approximations described in the report [10].

A somewhat different approach to approximation of multivariate data is studied in the current paper. Its main idea consists in projection of the set of data points from the original, higher-dimensional space onto a line section. Then, instead of the least squares minimization, the Galerkin minimization procedure is applied for finding the coefficients of approximation function. Certain gain in the computational efficiency can thus be attained, as the solution of an approximation problem is accomplished in a dimensionally reduced space. The description of the method is given in the two-dimensional setting, but it seems to be straight forward applicable also to more variables.

2. GALERKIN FIT

A set of scattered function-value data $F = \{F_j(\mathrm{x}_j) : \mathrm{x}_j \in \mathrm{D}, j=1\ldots\mathrm{n}\}$ is defined on a closed domain $\mathrm{D} \subseteq \mathrm{R}^d$. For simplicity of presentation, it is assumed throughout the paper that d=2. A local approximant $\zeta=\mathrm{p(x)a}$ to the data F is built by using a polynomial approximation basis $\mathrm{p}=[p_1,p_2,\ldots,p_\mathrm{m}]$, which is linear [1,x,y] when m=3, quadratic $[1,x,y,xy,x^2,y^2]$ if m=6 or it may be any other complete basis. The vector of coefficients a is to be established in the fitting procedure, which is outlined below.

First, the approximation errors are defined by $\Delta F_j = \mathrm{p}(\mathrm{x}_j)\mathrm{a} - F_j(\mathrm{x}_j)$ for each node in the original region D. Then, the nodes are projected onto a line section $\Lambda \subseteq \mathrm{R}^1$, where they are distributed equidistantly, as illustrated in Fig. 1.

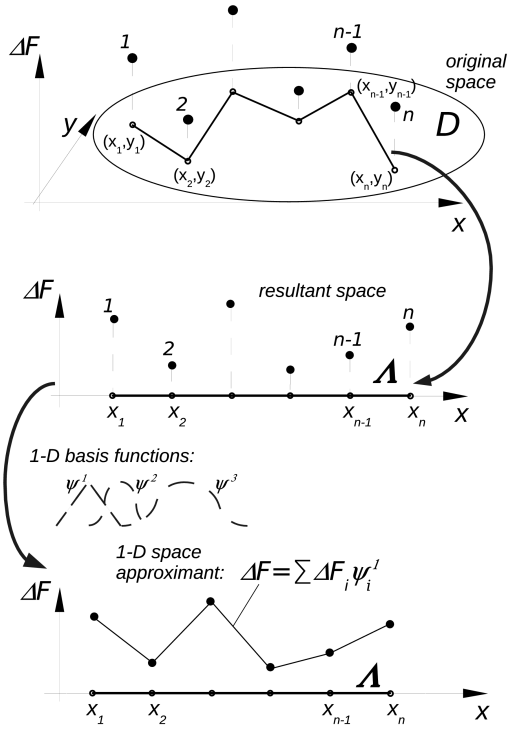

Fig. 1. The idea of approximation by using space dimensional reduction

The equidistant distribution of the nodes on Λ is an arbitrary assumption of the developed method, originating from the observation that only the nodal values themselves - and not the distances between the points - are quantities being unchanged in the projection.

An approximation subspace $V_n = \{\psi_j^b(x), j=1\ldots n\}$ is then associated with Λ, where the basis functions ψ_j^b are simple 'hat' functions if b=1 or are higher order polynomials. Now, the 'distribution' of the approximation error in the resultant one-dimensional subspace can be expressed in terms of the basis ψ_j^b as follows:

$$\Delta F = \sum_j [p(x_j)a - F_j]\psi_j^b(x), \tag{1}$$

where x={x,y} represents the original coordinates and x is the coordinate measured along Λ. Consequently, the Galerkin minimization is performed for each node i=1…n, according to the expression

$$\int_{\Lambda} \Delta F \cdot \psi_i^b dx = \int_{\Lambda} \sum_j [p(x_j)a - F_j]\psi_j^b(x) \cdot \psi_i^b(x) dx \tag{2}$$

and when usual transformations are done, a linear matrix equation is obtained, which contains the unknown coefficients a,

$$\mathbf{Ca} = b \quad . \tag{3}$$

In the above equation, the elements of matrix C are given by the expressions $C_{ik} = \sum A_{ij} p_k(x_j)$, where $A_{ij} = \int_{\Lambda} \psi_i^b \psi_j^b dx$, vector a contains m unknowns a_1, $a_2 \ldots a_m$ and the right-hand-side vector elements are defined by relations $b_k = \sum_j A_{kj} F_j$. It is clear, that the number of nodes should be greater or equal to the dimension of the approximation space ($n \geq m$). Therefore, the matrix C has dimension $n \times m$ and to solve the above equation, one can proceed as follows. If $n=m$ then a simple interpolation problem is solved. If $n=m+1$, the row corresponding to the central node can be added to each other row and then the equation (3) can be solved with a quadratic matrix C. If $n>m+1$, the above procedure can be repeated for the central node and the excessive rows, corresponding to the nodes which are most distant from the central one, can be summed up together to obtain an $m \times m$ matrix again.

Alternatively, the procedure can be presented in a form of projection, resulting in the following compact representation of the approximant:

$$\zeta = \mathbf{pa} = \mathbf{pC^{-1}}b = \mathbf{pC^{-1}AF} = \Phi F \quad , \tag{4}$$

or, in a more detailed version,

$$\zeta = \sum_k \varphi_k F_k \quad , \tag{5}$$

with the basis functions defined by

$$\varphi_k = \sum_{j=1}^{m} (C^{-1}A)_{kj} p_j \quad . \tag{6}$$

At this point, a short reference to the standard FEM, LS and MLS methods seems to be due. In each case, the approximation problem is defined by matrices C=AP, where $P=[p_1, p_2, \ldots, p_n]^T$, and $\Phi=p(C^{-1}A)$, but the matrix A has various compositions, depending on the method, which can be found, for example, in

the textbook [2]. Particularly, $A_{LS}=P^T$, $A_{MLS}=P^TW$, where W is a diagonal matrix containing weights and $A_{FEM}=I$ (an identity matrix). It follows from our previous considerations that A_{Gal} is a tridiagonal matrix, composed of triplets (1/6, 2/3, 1/6), if ψ_i^b are linear functions (b=1), and has greater bandwidth, if ψ_i^b are higher-order polynomials.

The above remarks can be concluded with the statement, rather commonly apprehended, that, if computational efficiency is considered, the FEM is the most competitive among the methods and the MLS is the least effective one. The Galerkin fit with its tridiagonal matrix A_{Gal} follows the FEM. Another order of precedence is most probably predicated when the methods are compared in terms of their approximation accuracy. This issue is studied further in the text, where the results of a number of numerical tests are presented.

3. GALERKIN FIT APPLIED FOR SOLVING A HEAT CONDUCTION EQUATION

An approach to derive an approximate solution to the unsteady heat conduction equation is studied below. The problem is defined in the spatio-temporal region $D \times \langle 0, t_{tot}\rangle$, where t_{tot} denotes the total computing time. With the temperature $T=T(x, t)$ as the main variable and with constant material properties μ, the governing equation for heat conduction, together with the boundary and initial conditions, is as follows:

$$\begin{aligned} \frac{\partial T}{\partial t} &= \mu \nabla^2 T \quad \text{in } \mathbf{D} \times \langle 0, t_{tot}\rangle, \\ T &= f(x,t) \quad \text{on } \partial\mathbf{D} \times (0, \underline{t}_{tot}\rangle, \\ T &= T_0 \quad \text{in } \mathbf{D} \times \{0\}, \end{aligned} \tag{7}$$

where ∂D denotes the boundary of the region D.

An approximated solution to the above differential equation can be obtained by coupling spatial discretization, performed with the developed method, with any recipe for temporal differentiation. In this exemplary application, the simplest algorithm suitable for performing comparison tests among the studied methods is chosen. Thus, the Euler time differencing algorithm yields the scheme

$$\tilde{T}_i^+ = \tilde{T}_i + \Delta t \cdot (\mu \nabla^2 T)_i \,, \tag{8}$$

where $\tilde{T}_i$ and $\tilde{T}_i^+$ denote the initial condition and the approximated solution at *i*-th node, respectively, and the term $\mu\nabla^2 T$ is expressed in the local basis defined by Eq. (6) as follows:

$$(\mu\nabla^2 T)_i = \mu\sum_k (\nabla^2\varphi_k)\tilde{T}_k\,, \tag{9}$$

where k=1, 2…i…n are indexes of the nodes surrounding, and include, the node i.

This way a point collocation method is obtained (cf. [2]), however, application of the weighted Galerkin formulation for the spatial approximation is also not precluded. Consequently, the Eq. (8) can serve for the comparison among the four above discussed methods in terms of their approximation quality. This issue is addressed in the next section.

4. TESTS AND RESULTS

4.1. Local approximation errors

In the first group of tests, the local Galerkin fit has been compared with the least squares (LS) and the moving least squares (MLS) methods. The comparison has been based on an interpolation example analyzed by Zienkiewicz [2], illustrated here in Fig. 2 with filled circles.

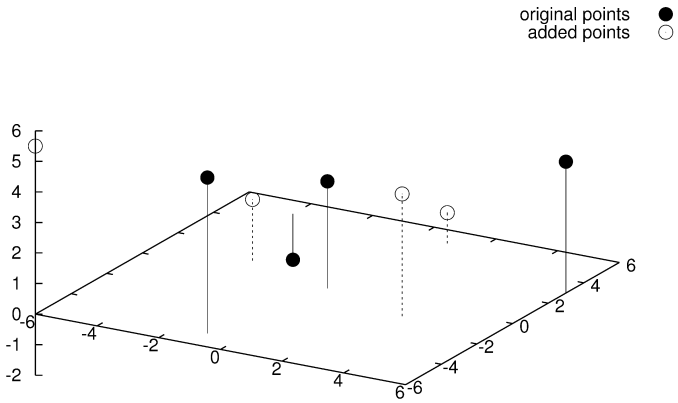

Fig. 2. Data set for the comparison test

Additional points are introduced for the purpose of the current study, to enable approximations to be done with the 2-nd order polynomials, and these points are

illustrated in Fig. 2 with the empty circles. An ill (singular) pattern of nodes is realized by relocating one of them to obtain four nodes alined.

Tables 1-2 show approximation errors computed for the singular and non-singular nodal patterns. The Galerkin methodology is performed by relying on the linear (ψ_i^1 = Gal) basis functions, illustrated in Fig. 1. In all the cases the quadratic approximation basis p is used, so m=6. Table 1 shows the results for the 6-node setup of nodes, i.e. for the interpolation case. The Galerkin fit is for that case more efficient than the LS and MLS techniques, however, for singular pattern of nodes, it gives worse results than its counterparts.

Tab. 1. Interpolation errors: 6 nodes

	normal setup		singular setup	
method	central node	mean	central node	mean
LS	0	0	1.053	6.473
MLS	0	0	0.264	2.499
Gal	0	0	-1.472	8.129

Tab. 2. Approximation errors: 8 nodes

	normal setup		singular setup	
method	central node	mean	central node	mean
LS	0.144	0.115	1.000	0.426
MLS	0.013	0.227	0.265	1.511
Gal	0.125	0.192	2.212	1.261

Table 2 itemizes the errors for the 8-node stencils. The superior performance of the MLS fit over all other methods is observed at the central node. However, if the mean error is studied, the best results are found for the LS approximation method, followed by Gal fit. A 7-node stencil has been studied also, but the results of Galerkin method in this case were more unsatisfactory.

4.2. Solution of the heat conduction equation

The above discussed methods are now applied for the solution of heat conduction problem described in the preceding section. The differential equation (7) has been solved under the initial condition $T(x,y,0)=sin(\pi x)+sin(\pi y)$ and with the boundary conditions $T(0,y,t)=T(1,y,t)=exp(-\mu\pi^2 t)\cdot sin(\pi y)$ and $T(x,0,t)=T(x,1,t)=exp(-\mu\pi^2 t)\cdot sin(\pi x)$. The analytical solution to this problem is given by the function $T(x,y,t)= exp(-\mu\pi^2 t)\cdot (sin(\pi x)+sin(\pi y))$.

The computations have been performed using regular and random distribution of nodes in the domain D, which is shown in Fig. 3. The random

distribution of the nodes is obtained by applying the following transformation to the regular nodes (x_i,y_i):

$$x_i' = x_i + r \cdot dx \cdot s\,, \tag{10a}$$

$$y_i' = y_i + r \cdot dy \cdot s\,, \tag{10b}$$

where r is a random number generated with C function drand48(), dx and dy are fractions (here 0.15) of the inter-nodal distances, and s=1 or s=-1, depending on the location of r within the interval $\langle 0,1\rangle$ subdivided into ten equal subintervals.

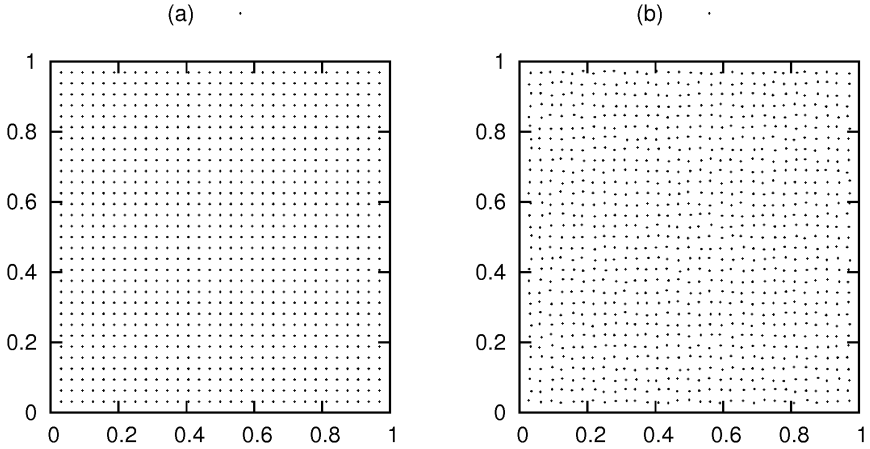

Fig. 3. Regular (a) and irregular (b) nodal arrangements for the heat conduction test

The convergence of the method (8) is illustrated, respectively, in Fig. 4 and in Fig. 5 in terms of the L_2-norm error. Each figure contains two plots, for the 6- and 8-node local approximation setups. The errors are plotted for the three above discussed approximation approaches. It should be mentioned that the approximation nodes are collected around each local center in an automatic manner. Searching algorithms from the ANN library [11] are used for that purpose.

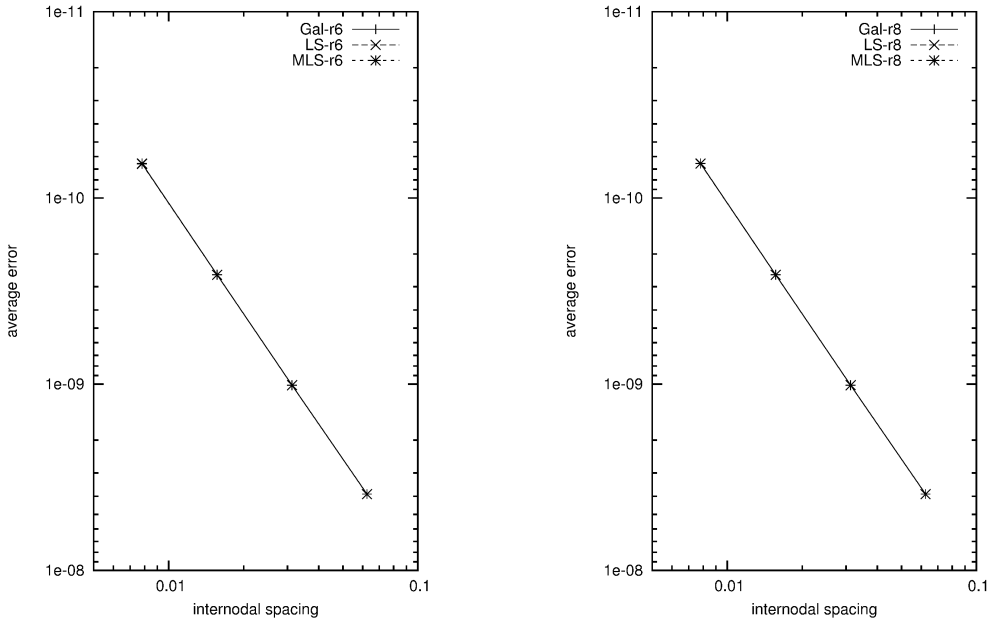

Fig. 4. Convergence plots of the average error for the regular arrangement of nodes

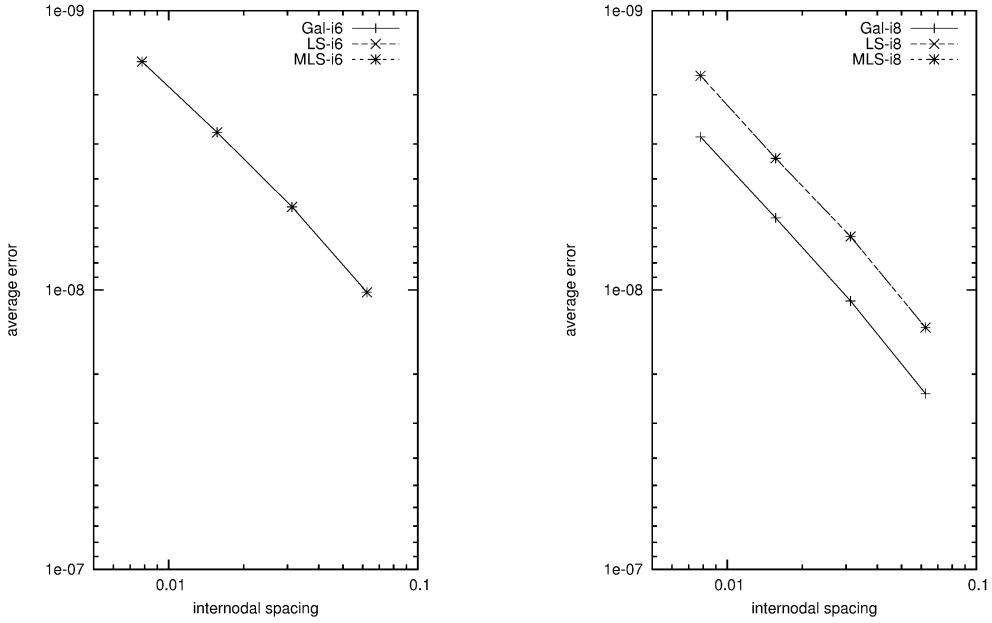

Fig. 5. Convergence plots of the average error for the irregular arrangement of nodes

A second-order convergence is attained uniformly by all the methods when the nodes are distributed regularly, see the curve inclination on the plots in Fig. 4. With the irregular nodal arrangement, the convergence deteriorates to about 1-st order, and the Gal approximations are a little less accurate then the least squares methods.

5. CONCLUSIONS

From the present study, the following conclusions can be drawn. The approximation method based on dimensional reduction and Galerkin minimization yields results comparable to the LS and MLS methods.

The developed method is computationally more efficient than the LS and MLS fits and attains similar accuracy, whether the nodes are distributed regularly or irregularly in the 2-D region, using 2-nd order polynomials and 6- or 8-node stencils.

The method has been successfully applied in a mesh-free, automatic, explicit solver of the unsteady heat conduction equation. It seems that the obtained results are encouraging to undertake further investigations in this area, including approximation in 3-D domains, other approximation bases and other applications.

REFERENCES

[1] MITCHELL A.R., WAIT R. (Eds.): The finite element method in partial differential equations. Wiley, Chichester, 1977.

[2] ZIENKIEWICZ O.C., TAYLOR R.L., ZHU J.Z. (Eds.): Finite Element Method - Its Basics and Fundamentals, KNovel Library, Elsevier, 2005.

[3] BELYTSCHKO T., KRONGAUZ Y., ORGAN D., FLEMING M., KRYSL P.: Meshless methods: an overview and recent developments. Comput. Methods Appl. Mech. Engrg., 139, 1996, 3-47.

[4] LIU G.R.: Mesh Free Methods: Moving Beyond the Finite Element Method. CRC Press, 2002.

[5] LANCASTER P., SALKAUSKAS K.: Surfaces generated by Moving Least Squares Methods. Math. Comp., 37, 1981, 141-158.

[6] WENDLAND H.: Local polynomial reproduction and moving least squares approximation. IMA J. Num. Anal., 21, 2001, 285-300.

[7] BREITKOPF P., RASSINEUX A., TOUZOT G., VILLON P.: Explicit form and efficient computation of MLS shape functions and their derivatives. Int. J. Numer. Meth. Engng, 48, 2000, 451-466.

[8] FASSHAUER G.E.: Matrix-free multilevel moving least squares methods, in: C.K. Chui, L.L. Schumaker, J. Stőckler (Eds.), Approximation Theory X: Wavelets, Splines and Applications. Vanderbilt University Press, 2002, 271-281.
[9] FASSHAUER G.E.: Towards approximate moving least squares approximation with irregularly spaced centers. Comput. Methods Appl. Mech. Engrg, 194, 2004,1231-1243.
[10] FASSHAUER G.E., ZHANG J.G.: Iterated approximate moving least squares approximation, in: V.M.A. Leitao, C. Alves and C.A. Duarte (Eds.), Advances in Meshfree Techniques. Springer, Netherlands, 2007.
[11] MOUNT D.M., ARYA S.: ANN: a library for approximate nearest neighbor searching. available from <http://www.cs.umd.edu/~mount/ANN>.